J. Fluid Mech. (1980), *vol.* 97, *part* 1, *pp.* 53–64
Printed in Great Britain

Penetration of fluid into a Hele–Shaw cell: the Saffman–Taylor experiment

By E. PITTS

Research Division, Kodak Limited, Headstone Drive, Harrow, Middlesex HA1 4TY, England

(Received 7 June 1978 and in revised form 25 May 1979)

A theoretical account is given of experiments performed by Saffman & Taylor (1958) in which a fluid drives a liquid out of a long straight channel of very small thickness formed between two parallel sheets sealed at the edges. The penetrating fluid forms a long finger, whose sides are parallel to the edges of the channel, and which has a rounded tip, which advances with unaltered shape at a constant speed U. The theory correctly predicts the shape of the finger as a function of the ratio λ = (asymptotic width of finger)/(width of channel) and gives the relation between λ and U, which is in good agreement with experiment. In particular it shows that, as U increases from zero to infinity, λ steadily decreases from 1 to 0·5.

1. Introduction

During the course of an investigation concerning the penetration of a fluid into a porous medium, Saffman & Taylor (1958, 1959) carried out a remarkable experiment, for which (as they pointed out) their theoretical account was in several respects incomplete. It appears (see Wooding & Morel-Seytoux 1976) that the experiments still await explanation. It is the object of this paper to give a theoretical basis for their observations.

Saffman & Taylor constructed a Hele–Shaw cell of dimensions $91 \times 2{\cdot}54 \times 0{\cdot}08$ cm which they filled with oil. By suitable means they applied water under pressure at one end of the cell, as if to drive out the oil ahead of the water. Whilst some oil was swept out in this fashion, the water also penetrated the cell as a finger having a rounded advancing tip and long sides parallel to the edges of the channel (see figure 1). Photographs were taken showing the shape of the meniscus profile between the oil and the water, viewed through the narrow thickness of the cell. Essentially similar results were found when other pairs of fluids were used. From their measurements, Saffman & Taylor found that the asymptotic width of the finger far from the advancing tip was never less than half the width of the channel; for very slow rates of advance, almost all the cell contents were swept out, and as the speed of the finger was increased (by increasing the pressure applied to the penetrating fluid) the width steadily approached the limiting value of one half of the channel width.

In the next section, an abbreviated version of the Saffman–Taylor theory is given together with a description of the way in which the experimental results diverge from its predictions. Subsequent sections give an account of the new treatment of the problem and the comparison with experiments which confirm it. Throughout the

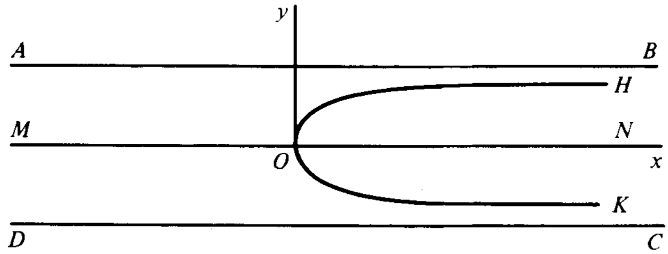

FIGURE 1. The co-ordinate system. HOK is the outline of the finger moving to the left with speed U.

paper the term 'profile' will refer to the strongly curved region of the interface between the fluids, which is observed as the edge of the finger.

2. The Saffman–Taylor theory

Figure 1 shows the typical appearance of the finger of the advancing fluid (defined by the profile HOK) moving from right to left with speed U at the tip O, into the cell whose edges are AB and CD and whose narrowest dimension lies perpendicular to the plane of the paper. It was found that the finger is symmetrical with respect to the centre-line MON. Saffman & Taylor discuss the problem in conventional Hele–Shaw terms; that is, they regard the motion as two-dimensional and consider velocities which describe the motion averaged across the thickness of the cell. With the origin of co-ordinates placed as shown in figure 1, the mean velocity is described by the components $u(x,y)$ and $v(x,y)$, parallel respectively to the x and y axes, which are fixed to the channel. These velocities are related to a potential function ϕ and a stream function ψ by the usual equations

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y},$$

$$v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

Far ahead of the finger, the mean velocity is taken to be V. Saffman & Taylor define the ratio λ equal to HK/BC, that is the asymptotic width of the finger divided by the width of the channel. They choose a unit of length equal to half the channel width. Noting the experimental fact that the velocity in the liquid between the meniscus and the edge of the channel is zero very far from the tip, it follows from continuity that

$$V = \lambda U. \tag{2.1}$$

Saffman & Taylor expand the complex variable

$$z = x + iy$$

as a series of exponential functions of the complex potential

$$\omega = \phi + i\psi$$

of the form (for the co-ordinate system of figure 1)

$$z = -\frac{\omega}{V} + \sum_{n=0}^{\infty} A_n \exp(-n\pi\omega/V). \tag{2.2}$$

The imaginary part of this equation is

$$y = -\psi/V - \sum_{n=1}^{\infty} A_n \exp(-n\pi\phi/V) \sin(n\pi\psi/V), \tag{2.3}$$

and Saffman & Taylor argue that, on the meniscus, pressure is constant, i.e. ϕ is a constant (zero). Also,

$$y = -\psi/U, \tag{2.4}$$

which expresses the fact that the normal liquid velocity must equal the normal component arising from the motion of the finger, which has a profile which does not change shape with time. On AB and CD, which are streamlines, ψ is equal to $\mp V$ respectively. Substitution in (2.3) gives a Fourier series

$$-\psi/U = -\psi/V - \sum_{n=1}^{\infty} A_n \sin(n\pi\psi/V), \quad -V \leqslant \psi \leqslant V,$$

from which

$$\pi n A_n = 2(1-\lambda)(-1)^n.$$

Hence from (2.2) and the condition $\omega = 0$ when $z = 0$

$$z = -\frac{\omega}{V} - \frac{2(1-\lambda)}{\pi} \ln\left(\frac{1+e^{-\pi\omega/V}}{2}\right). \tag{2.5}$$

The parametric equation of the meniscus is obtained by putting $\phi = 0$ and $\psi = -Uy$ in this equation, so that from the real part, after straightforward simplifications,

$$\exp\left[\frac{\pi x}{2(1-\lambda)}\right] \cos\left(\frac{\pi y}{2\lambda}\right) = 1. \tag{2.6}$$

Saffman & Taylor showed that the result (2.6) agrees very well with the observed profile when λ is equal to 0·5, which is the limiting value for high velocities. However, they point out that, for other values of λ, the profile deviates steadily from that observed. Indeed as λ approaches unity, the profile (2.6) tends to a straight line at right angles to the abscissa, whereas the actual profile remains strongly curved.

To the problem of the relation between U and λ, Saffman & Taylor were unable to give any answer. Their analysis (summarized above) gave no indication of any restriction on λ to values exceeding 0·5, nor did it suggest any physical relation between λ and U.

3. The equation of the meniscus profile

From the failure of the analysis outlined in the preceding section to describe the general behaviour of the profile, it seems clear that the physical conditions imposed by Saffman & Taylor have to be abandoned. In searching for a replacement, methods based on Fourier series, hodograph techniques and the superposition of singularities were explored without any appropriate criterion becoming apparent.

Further progress became possible after careful observation of the flow of liquid very close to the profile. As the profile advances, thin sheets of liquid are left on the surfaces of the cell. When very small particles of dirt appeared in these thin liquid regions close to the profile, it appeared that the particles moved along a normal and either eventually crossed the profile, entering the liquid ahead of it, or were left behind on the cell surface. Larger particles present in this region upstream, but very near the profile, did not cross the profile into the thin sheets of liquid but appeared to be pushed aside, again along a normal to the profile. In both cases the extent of tangential motion was difficult to judge, but seemed to be small very close to the profile. For particles which were in the liquid some distance away from the profile, the main component of velocity was parallel to the axis of the cell.

The velocity distribution in the liquid region near the strongly curved interface is obviously very complicated and different from that assumed in conventional Hele–Shaw flow. However, the observations suggest that in this region some streamlines are present which cross the profile at right angles. In a plane perpendicular to the cell surfaces, the meniscus has an approximately semicircular section and therefore, as it moves forward, the liquid between it and the surface is contained within a narrowing channel which finally becomes the thin liquid sheet on the cell surface. The pressure of the gas causing the motion of the profile acts along the normal in the plane of the cell, and it would thus be expected that (relative to the moving profile) the liquid adhering to the surface as the profile moves over it could instantaneously be regarded as moving along the normal, with no tangential velocity component. At every point on the expanding profile, there is in effect a viscous drag acting along a normal across it, and it is this feature which is decisively different from the usual examples of Hele–Shaw flow round obstacles.

It appears difficult to find a completely satisfactory argument by which the choice of boundary conditions to be applied in the Hele–Shaw treatment of this problem can be justified, principally because the behaviour of the fluid near the profile is so complicated in detail. However, a result in striking agreement with experiment has been found, namely that the profile is such that, if R is the radius of curvature at a point on the curve where the gradient is $\tan\theta$, then

$$R\sin\theta = \text{constant}. \tag{3.1}$$

In spite of considerable effort, a satisfactory justification of this relation has not been found.

Equation (3.1) can be written

$$q_x = -aq(1+q), \tag{3.2}$$

where

$$q = y_x^2$$

and the suffix denotes differentiation with respect to x, and a is a positive constant. This equation may be integrated with the result

$$q = (e^{ax}-1)^{-1}$$

so that, as x tends to infinity, q tends to zero. A further integration gives

$$\exp(-\tfrac{1}{2}ax) = \cos(\tfrac{1}{2}ay+b).$$

The constant b must be zero since y is zero when x is zero. Also, when x tends to infinity, the value of $2y$ must tend to the limiting value of the width of the finger, namely 2λ. This is so if a is equal to π/λ. We therefore obtain for the profile the equation

$$\exp\left(\frac{\pi x}{2\lambda}\right)\cos\left(\frac{\pi y}{2\lambda}\right) = 1. \tag{3.3}$$

In § 6 we shall show that this equation reproduces very closely the experimental results.

Two further remarks may be made here. It will be noticed that (3.3) differs from the result (2.6) in that λ replaces $(1-\lambda)$ as the divisor of πx. Secondly, if we define new variables

$$\xi = \frac{\pi x}{2\lambda}, \quad \eta = \frac{\pi y}{2\lambda},$$

the equation for the profile is

$$e^{\xi}\cos\eta = 1 \tag{3.4}$$

and the edges of the channel occur at $\eta = \pm\pi/2\lambda$. In these co-ordinates, all the profiles are reduced to a single curve (3.4) irrespective of the position of the edges of the channel. All the observed profiles can be reduced to a single curve by uniform magnification of the axes.

4. The velocity field

To find the velocities round the finger, it will be convenient to regard the profile as at rest with respect to the axes and the upper and lower faces of the cell in motion from left to right in figure 1 with velocity U and modify the Hele–Shaw theory accordingly. If $s_x(x,y,z)$ and $s_y(x,y,z)$ are components of velocity in the liquid, and u and v as before are the corresponding mean velocities, with the customary assumption of quadratic dependence on z for s_x and s_y, we find

$$s_x = U + 6(U-u)(4z^2 - z_0^2)/4z_0^2,$$
$$s_y = -6v(4z^2 - z_0^2)/4z_0^2,$$

where the total thickness of the cell is z_0. Substitution of these expressions in the hydrodynamic equations and the assumption that z_0 is extremely small compared with physically significant lengths in the x, y plane gives the equations for the pressure p:

$$\frac{\partial p}{\partial x} = 12\mu(U-u)/z_0^2,$$

$$\frac{\partial p}{\partial y} = -12\mu v/z_0^2,$$

where μ is the viscosity of the liquid. The relation between the pressure and the potential function ϕ is thus

$$p = 12\mu(Ux - \phi)/z_0^2. \tag{4.1}$$

Since the finger is at rest, the liquid in the region between H and K very far from the tip will be moving to the right with velocity U, whilst at large negative values of

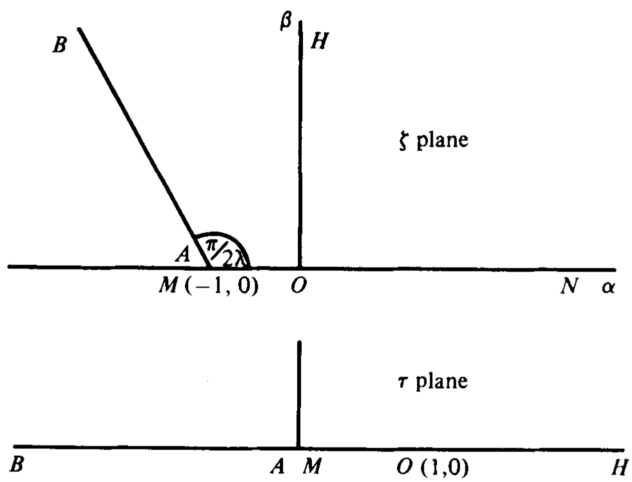

FIGURE 2. The ζ and τ planes: the lettering corresponds to that in figure 1.

x it will be moving to the right with velocity $U-V$, i.e. $U(1-\lambda)$. We have to find ϕ and ψ which satisfy these conditions, together with the condition that ψ is zero on the profile given by (3.3); this is the condition that no liquid crosses the interface.

The key to the solution of the problem by conformal mapping was eventually seen to be the construction of a transformation which mapped the liquid region $BAMOH$ in figure 1 on to a half-plane. This is accomplished in the following way. Consider a complex variable

$$\zeta = \alpha + i\beta \tag{4.2}$$

defined by the equation

$$\exp\left(\frac{\pi z}{2\lambda}\right) = 1+\zeta. \tag{4.3}$$

The real and imaginary parts of this equation are

$$\exp\left(\frac{\pi x}{2\lambda}\right)\cos\left(\frac{\pi y}{2\lambda}\right) = 1+\alpha, \tag{4.4}$$

$$\exp\left(\frac{\pi x}{2\lambda}\right)\sin\left(\frac{\pi y}{2\lambda}\right) = \beta. \tag{4.5}$$

Hence the profile (3.3) corresponds to $\alpha = 0$, that is the imaginary axis in the ζ plane (see figure 2). The origin O in the x, y plane is also the origin in the ζ plane, and the points A and M at $-\infty$ coalesce in the point $(-1, 0)$. On AB in figure 1, $y = 1$ and evidently from (4.4) and (4.5)

$$\beta/(1+\alpha) = \tan(\pi/2\lambda).$$

Thus in figure 2 the line BA with angle BAO equal to $(\pi/2\lambda)$ corresponds to the edge of the channel. The region $BAMOH$ occupied by liquid is thus mapped into the polygon $BAOH$ in the ζ plane. By means of a Schwarz–Christoffel transformation this may be mapped into the upper half of the τ plane with A and M at the origin and O at $\tau = 1$ (see figure 2). The appropriate transformation is

$$\zeta = k\int_0^{\tau} \frac{ds}{s^{(2\lambda-1)/2\lambda}(1-s)^{\frac{1}{2}}} + l$$

$$\equiv kB\left(\frac{1}{2\lambda}, \tfrac{1}{2}, \tau\right) + l, \tag{4.6}$$

where k and l are constants and B is the incomplete beta function. Since when τ is zero ζ is equal to -1, we find $l = -1$. Also, when $\tau = 1$, ζ is zero and so

$$k(\lambda) = \left[B\left(\frac{1}{2\lambda}, \tfrac{1}{2}, 1\right)\right]^{-1} = \Gamma\left(\frac{1+\lambda}{2\lambda}\right)\left[\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2\lambda}\right)\right]^{-1} \tag{4.7}$$

(from the properties of the beta function). Combining these results we obtain

$$\exp\left(\frac{\pi z}{2\lambda}\right) = k(\lambda)\, B\left(\frac{1}{2\lambda}, \tfrac{1}{2}, \tau\right). \tag{4.8}$$

We now require a mapping of the complex potential $\omega = \phi + i\psi$ on to the τ half-plane. This is easily found, since on MOH the stream function ψ is zero, while on BA ψ is evidently equal to $U(1-\lambda)$. If we take ϕ equal to zero at O, then the appropriate transformation is

$$\tau = \exp\left[\frac{\pi\omega}{U(1-\lambda)}\right]. \tag{4.9}$$

This equation and (4.8) give the required relation between ω and z from which the features of the flow can be derived.

A consideration of the velocities gives a result required later. By differentiation of (4.8) with respect to z and use of (4.9) we find

$$u - iv = U(1-\lambda)\,(2\lambda k)^{-1}\,\tau^{-1/2\lambda}(1-\tau)^{\frac{1}{2}}\,e^{\pi z/2\lambda}. \tag{4.10}$$

Now, as x tends to infinity, u tends to U and v to zero; hence asymptotically we must have

$$\omega \sim Uz + c, \tag{4.11}$$

where c is a constant and the remaining terms are vanishingly small. When this is substituted in (4.10), we find from the leading term

$$(-1)^{\frac{1}{2}}\,(1-\lambda)\,(2\lambda k)^{-1}\,e^{-\pi c/2\lambda U} = 1. \tag{4.12}$$

Hence

$$c = \frac{2\lambda U}{\pi}\ln\left(\frac{1-\lambda}{2\lambda k}\right) - i\lambda U, \tag{4.13}$$

a result needed in the evaluation of the pressure.

5. The λ, U relation

To complete the solution of the problem we have to relate the velocities to the driving force in the physical problem, which is the pressure in the advancing finger. Saffman & Taylor show that experimental conditions are such that the pressure p_1

in the penetrating fluid can be regarded as practically uniform everywhere; this is obviously the case if the fluid is air. We now apply this condition to the theory of § 4.

The pressure p_∞ in the liquid near H far from the origin will be less than p_1 owing to the action of surface tension γ. We assume that the effective radius of curvature (which is solely in the y, z plane) is simply $\frac{1}{2}z_0$, corresponding to a meniscus with semi-circular cross-section. Thus

$$p_\infty = p_1 - 2\gamma/z_0. \tag{5.1}$$

At the origin, there are two principal radii of curvature to be considered. From the result (3.3) we find that the radius in the x, y plane is

$$r_1 = 2\lambda/\pi. \tag{5.2}$$

The radius of curvature r_2 in the x, z plane is of order $\frac{1}{2}z_0$, and, since the surfaces of the cell are moving with velocity U in the x direction above the stationary tip of the finger, it is reasonable to expect that viscous drag will reduce r_2 from its value $\frac{1}{2}z_0$ expected in static conditions. We therefore put

$$r_2 = z_0/2m, \tag{5.3}$$

with $m > 1$.

If we ignore the contribution to normal stress arising from the viscosity and the velocity gradient (this can be shown to be small compared with the term due to curvature in the x, y plane) we obtain for the pressure p_0 at the origin

$$p_0 = p_1 - \gamma(r_1^{-1} + r_2^{-1}). \tag{5.4}$$

By subtraction we obtain an expression for the pressure drop

$$p_\infty - p_0 = 2\gamma(m-1)/z_0 + \pi\gamma/2\lambda. \tag{5.5}$$

For the apparatus used by Saffman & Taylor, the ratio of the terms on the right-hand side of this equation is

$$\frac{\pi z_0}{4\lambda(m-1)} \doteqdot \frac{0{\cdot}05}{\lambda(m-1)}.$$

Unless m is very close to unity (which seems unlikely from the considerations which are discussed in § 7), this ratio is small. Moreover, this ratio is proportional to z_0, which is ideally of negligible magnitude. This suggests that it may be sufficient to write as an approximation

$$p_\infty - p_0 = 2\gamma(m-1)/z_0. \tag{5.6}$$

From (4.1) we then obtain

$$2\gamma(m-1)/z_0 = 12\mu[(Ux-\phi)_\infty - (Ux-\phi)_0]/z_0^2. \tag{5.7}$$

The real parts of (4.11) and (4.13) give

$$(Ux-\phi)_\infty = -\frac{2\lambda U}{\pi}\ln\left(\frac{1-\lambda}{2\lambda k}\right),$$

and $(Ux-\phi)_0$ is by definition zero. If we define

$$F(\lambda) = \left[\lambda \ln\left(\frac{2\lambda k}{1-\lambda}\right)\right]^{-1},$$

λ	$F(\lambda)$	λ	$F(\lambda)$
0·50	∞	0·74	1·695
0·52	34·403	0·78	1·305
0·54	16·417	0·82	1·016
0·56	10·444	0·86	0·793
0·58	7·477	0·90	0·611
0·62	4·541	0·94	0·454
0·66	3·099	0·98	0·295
0·70	2·250	1·00	0

TABLE 1.

substitution in (5.7) gives

$$F(\lambda) = \frac{12(\mu U/\gamma)}{\pi z_0(m-1)}. \tag{5.8}$$

(It should be recalled that z_0 is the ratio of the thickness of the cell to half its width, which is adopted as the unit of length.)

We now assume that, to a good approximation, m is independent of U. Then (5.8) relates λ and the dimensionless group $(\mu U/\gamma)$, which is the result we require. It will be seen from the values of $F(\lambda)$ in table 1 that, when λ is 0·5, U is infinite and, when λ is 1, U is zero. These are the limiting values experimentally observed, and we now turn to a detailed comparison with experiment.

6. Experimental results and comparison with theory

In order to show whether (3.3) correctly describes the profile, photographs of its shape were required. Accordingly, apparatus with the same dimensions as that used by Saffman & Taylor (1958) was constructed. This proved to be far from straightforward, since clamping the plastic sheets (forming the surfaces of the cell) on to the rubber spacers (forming the edges) at intervals introduced periodic slight variations in thickness of the cell which revealed themselves as regular variations in the width of the finger. Eventually these difficulties were overcome and profiles were obtained having very long sides parallel to the edges of the channel, like those illustrated by Saffman & Taylor.

The channel was filled with a corn oil (specific gravity 0·916, viscosity 68·55 centipoise, surface tension 27·7 dynes/cm) which was dyed blue for ease of viewing. This was displaced in the cell by air under pressure. Photographs were taken of the profile and enlargements produced, from which accurate measurements were easily made.

In figures 3 (plate 1) and 4 the experimental results are compared with theory. Figure 3 is a composite photograph made in the following way. An enlargement of the photograph of the profile of the widest finger ($\lambda = 0{\cdot}77$) was made, and then enlargements of profiles for $\lambda = 0{\cdot}67$ and 0·54 were made such that for each the limiting width of the finger was equal to that in the photograph for $\lambda = 0{\cdot}77$. A plot, marked by crosses alone, of the theoretical expression (3.3) was then made, again with the same limiting width. These four positive transparencies were then brought into register, and the positive shown in figure 3 was made. It will be seen that the

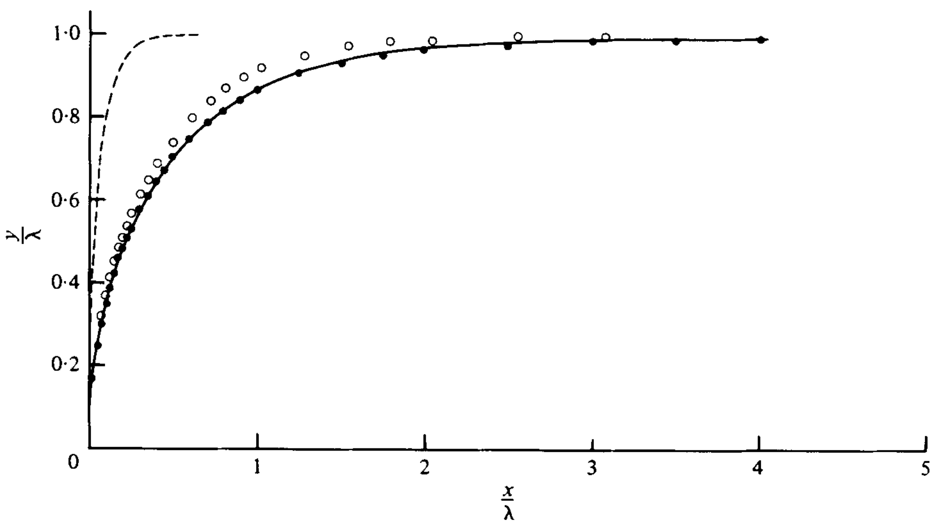

FIGURE 4. Profile calculated from (3.3), ——. Observed profiles: ●, $\lambda = 0{\cdot}54, 0{\cdot}67, 0{\cdot}77$; ○, $\lambda = 0{\cdot}88$. Curve calculated from (2.6) for $\lambda = 0{\cdot}88$, ----.

three experimental curves are virtually coincident, and the crosses lie over the common curve. Close examination reveals slight discrepancies in the positions of the three curves. Also, the different widths of the channel, arising from the three different degrees of enlargements, are visible. Measurement shows that these widths are proportional to $(0{\cdot}54)^{-1}$, $(0{\cdot}67)^{-1}$ and $(0{\cdot}77)^{-1}$ as pointed out at the end of § 3.

These results have been replotted in figure 4, in which the position of the three virtually coincident curves has been indicated by single points to avoid confusion. The solid curve is the theoretical expression, and is in excellent agreement with the experimental observations. However, for $\lambda = 0{\cdot}88$, which corresponds to a very much slower speed of advance of the finger, there is a perceptible departure from the theory. This is undoubtedly partly due to the changes which occur very slowly in the thin film of liquid left on the surfaces of the cell after the passage of the tip; the film gradually thins near the profile and thickens slightly towards the centre-line of the cell. Similar drainage effects were noticed by Saffman & Taylor (1958). Also, for values of λ approaching unity, the requirement that cell thickness is negligible compared to the width of the region occupied by liquid is not satisfied far downstream where the finger almost fills the cell. For comparison, the dotted curve shows the prediction for $\lambda = 0{\cdot}88$ derived from their result (2.6).

The relation between λ and U given by (5.8), with the assumption that m is constant, has been compared with the experimental results given by Saffman & Taylor (1958). In addition, a few observations were made as a check on their data. All these results have been used in figure 5, where the experimentally observed value of $(\mu U/\gamma)$ corresponding to a particular value of λ is plotted against the theoretical value of $F(\lambda)$. It will be seen that the results lie near a straight line, which supports the hypothesis that m is effectively constant. The gradient of the line in figure 5 is $2{\cdot}36$, and for the apparatus the slope predicted by (5.8) is $0{\cdot}606/(m-1)$ from which it follows that the appropriate value of m is $1{\cdot}26$. The scatter in results is greatest for large

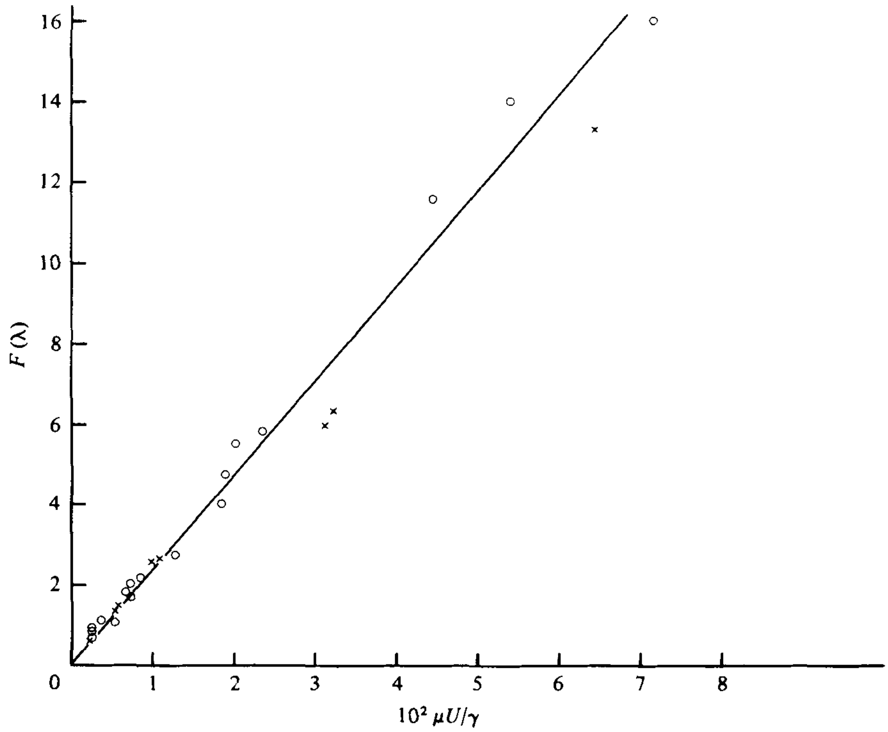

FIGURE 5. The relation (5.8) between $F(\lambda)$ and U compared with experiment: ○, Saffman–Taylor results; ×, new observations.

values of U, when λ approaches 0·5 and F is then a rapidly varying function of λ, so that small errors in λ correspond to large changes in F.

7. Discussion

This account of the Saffman–Taylor experiment rests on two crucial arguments neither of which is self-evident. The first is equation (3.1) which expresses the constancy of $R \sin \theta$ along the profile, and for which no satisfactory explanation has been found.

The second important hypothesis is the assumed constancy of m, which expresses the change in the physically dominating curvature of the meniscus at the tip from its value $(2/z_0)$ far downstream. In support of this assumption, the following comments may be put forward. Far from the tip, viscous drag due to motion of the cell surfaces acts in a direction perpendicular to the plane containing the maximum curvature. At the tip, on the axis of symmetry, the viscous drag is parallel to this plane and we should expect a marked change in curvature there. It seems plausible to expect that the difference between no motion at the tip (curvature $2/z_0$) and motion, however slight, which leaves a layer of liquid on the cell surfaces (when the curvature is $2m/z_0$) will be substantial. In other words, it seems likely that m does not tend to unity as U decreases towards zero. How m might change quantitatively as U increases is less obvious, but it seems very reasonable to expect that, for an appreciable range of velocities, m would change very little. Whatever the value of these qualitative considerations, it certainly appears that the results in figure 5 indicate that m is approximately constant.

A further comment may be made about the relationship between m and U. As an approximation, we could regard the shape of the cross-section of the advancing finger in the plane at right angles to the surface of the cell as similar to that viewed from above through the surfaces. The expression (5.2) gives the value $2\lambda/\pi$ for the radius of curvature in the plane of the surface, and we may imagine a similar relation to hold in the other principal plane of curvature. Since in this plane λ is very close to unity, this would suggest that the radius of curvature is $(2/\pi)(\frac{1}{2}z_0)$, that is, m equals $\frac{1}{2}\pi$. Whilst this estimate is very crude, it gives some support to the assumption that m differs appreciably from unity.

I am especially grateful to my colleague Mr G. C. Terry whose skill and patience overcame the considerable difficulties encountered in the construction of the apparatus, and who helped in taking measurements and many photographs. My attention was drawn to this interesting problem by Professor Saffman, to whom I should like to express my acknowledgements. I am also indebted to the referees, who pointed out the inadequacy of the physical arguments which had suggested to me that $R\sin\theta$ is constant on the profile (§ 3).

REFERENCES

SAFFMAN, P. G. & TAYLOR, G. I. 1958 *Proc. Roy. Soc.* A **245**, 312.

SAFFMAN, P. G. & TAYLOR, G. I. 1959 *Quart. J. Mech. Appl. Math.* **12**, 265.

WOODING, R. A. & MOREL-SEYTOUX, H. J. 1976 *Ann. Rev. Fluid Mech.* **8**, 233.

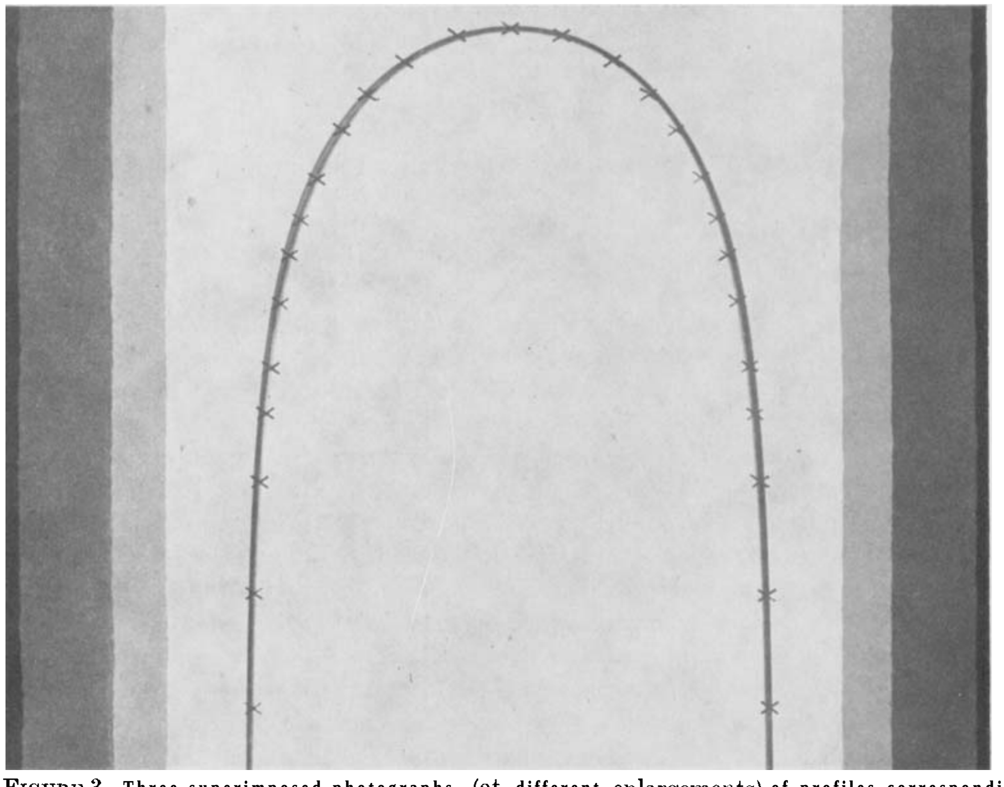

FIGURE 3. Three superimposed photographs (at different enlargements) of profiles corresponding to $\lambda = 0{\cdot}54$, $0{\cdot}67$ and $0{\cdot}77$. The crosses correspond to equation (3.3).